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LETTER TO THE EDITOR

A note on a representation of SO(3, 2)

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Abstract. The objective of this letter is to call attention to a class of special functions which arise from a representation of the SO(3, 2) group that is induced by the representation of its maximal compact subgroup SO(3)⊗SO(2). We give here the defining differential equations for these functions. A straightforward generalisation of this procedure will be applicable in the SO(p, q) case.

The de Sitter group, SO(3, 2), has received a considerable amount of interest due to its applications in elementary particle physics. The Majorana representation, which uses the analytic nature of the group actions, is of little use in this respect. Other representations, due to Fischer *et al* (1966), are constructed inductively from a chain of subgroups:

$$\mathrm{SO}(3,2) \supset \mathrm{SO}(2,2) \supset \mathrm{SO}(2,1) \supset \mathrm{SO}(2)$$

or

$$\mathrm{SO}(3,2) \supset \mathrm{SO}(3,1) \supset \mathrm{SO}(3) \supset \mathrm{SO}(2). \tag{1}$$

There, the representation spaces are the spaces of square integrable functions on the spaces $X_\pm = \{x \in R^5 | [x, x] = \pm 1\}$. [,] denotes the pseudo-metric $(+++--)$ and the irreducible spaces are further decomposed as irreducible spaces of the representations of the subgroups down the chain. Their inductive construction, however, does not include the case

$$\mathrm{SO}(3,2) \supset \mathrm{SO}(3)\otimes\mathrm{SO}(2) \supset \mathrm{SO}(3) \supset \mathrm{SO}(2). \tag{2}$$

This chain is, from a group theoretical viewpoint, particularly important since SO(3)⊗SO(2) is the maximal compact subgroup of SO(3, 2), from which we get the Iwasawa decomposition. A representation respecting this chain is available, according to the general theory on semi-simple Lie groups established by Harish-Chandra (see Helgason 1984). The representation space consists of functions on the coset space SO(3, 2)/SO(3)⊗SO(2), which is quite large (six-dimensional) as compared with the spaces considered by Fischer *et al.* Here we present a representation with respect to this chain and a class of special functions arise from such a representation, working within the space X_+. Our work closely follows the treatment of Gel'fand *et al* (1966) on the SO(3, 1) group, which is similar to the Fourier–Radon transform technique of Helgason. This general theory does not apply in our case since the dimension is not correct.

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Let $\Sigma=\{y\in R^5|[y,y]=0\}$, on which the SO(3, 2) group acts naturally. We then introduce parametrisation on Σ by

$$y=\sigma\begin{bmatrix}\sin\theta'\cos\phi'\\ \sin\theta'\sin\phi'\\ \cos\theta'\\ \cos\Omega'\\ \sin\Omega'\end{bmatrix}\qquad \sigma\geqslant 0. \tag{3}$$

A group invariant measure on Σ is $\sigma^2\,\mathrm{d}\sigma\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi'\,\mathrm{d}\Omega'$. The space $L^2(\Sigma)$ becomes a carrier space for the SO(3, 2) representation via

$$g(f(y))=f(g^{-1}(y))\qquad g\in \mathrm{SO}(3,2), f\in L^2(\Sigma). \tag{4}$$

An irreducible subspace consists of homogeneous functions of the same degree. Decomposition into irreducible components is given by

$$f(y)\to F(y,\mu)=\int_0^\infty f(ty)t^{-\mathrm{i}\mu+1/2}\,\mathrm{d}t. \tag{5}$$

Here the integral is well defined if we assume f is well behaved at 0. So to be mathematically precise, we should start with square integrable functions bounded at 0 as a dense subset in the Hilbert space $L^2(\Sigma)$. One checks that F is homogeneous of degree $\mathrm{i}\mu-\frac{3}{2}$, so F is determined by its function value on $S^2\times S^1$ in Σ. F is further decomposed as

$$F(y,\mu)=\sigma^{\mathrm{i}\mu-3/2}\Sigma c_{klm}(\mu)Y_{lm}(\theta',\phi')\exp(\mathrm{i}k\Omega'). \tag{6}$$

Here $Y_{lm}(\theta',\phi')$ and $\exp(\mathrm{i}k\Omega')$ are the spherical harmonics (eigenfunction of the Laplacian) on S^2 and S^1, respectively. The SO(3)$\otimes$SO(2) action leaves the indices l and k fixed. f can be recovered from F via

$$f(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(y,\mu)\,\mathrm{d}\mu. \tag{7}$$

Let $X=\{x\in R^5|[x,x]=1\}$ as before, parametrised as

$$x=\begin{bmatrix}\cosh\alpha\sin\theta\cos\phi\\ \cosh\alpha\sin\theta\sin\phi\\ \cosh\alpha\cos\theta\\ \sinh\alpha\cos\Omega\\ \sinh\alpha\sin\Omega\end{bmatrix}. \tag{8}$$

Let $\mathrm{d}x$ denote the SO(3, 2) invariant measure

$$\mathrm{d}x=\cosh^2\alpha\sinh\alpha\sin\theta\,\mathrm{d}\alpha\,d\theta\,\mathrm{d}\phi\,\mathrm{d}\Omega.$$

Then $L^2(X)$ is an SO(3, 2) representation space in the usual manner. Define an SO(3, 2) action-preserving transformation

$$\begin{aligned}&L^2(X)\to L^2(\Sigma)\\ &h(x)\to f(y)=\int h(x)\delta([x,y]-1)\,\mathrm{d}x.\end{aligned} \tag{9}$$

Here δ is the Dirac delta function. Note that if $h(x)$ has bounded support, $f(0)=0$, thus bounded, and equation (5) is well defined. Functions of bounded support are dense in $L^2(X)$. So (5) and (9) are defined on the appropriate dense subsets and extend to the whole space using standard mathematical arguments.

The irreducible subspace decomposition on $L^2(\Sigma)$ induces one on $L^2(x)$, using the fact that δ is homogeneous of degree -1. One can show that

$$\begin{aligned} F(y,\mu) &= \int h(x)\delta([x,ty]-1)\, t^{-\mathrm{i}\mu+1/2}\,\mathrm{d}t\,\mathrm{d}x \\ &= \int h(x)[x,y]^{\mathrm{i}\mu-3/2}\,\mathrm{d}x. \end{aligned} \tag{10}$$

Fix $y\in\Sigma$ and then the functions $[x,y]^{\mathrm{i}\mu-3/2}$ as a function on X can be viewed as plane waves and (10) as plane wave decompositions, analogous to $\exp(\mathrm{i}kx)$ in the Euclidean space since they are eigenfunctions of the Laplace-Beltrami operator on X. If we parametrise R^5 by rx, $r>0$, x as in (8), then the Laplace-Beltrami operator (with respect to the metric $+++--$) is given by

$$\Box_{R^5}=\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Box_X \tag{11}$$

where $\Box_X$ is the Laplace-Beltrami operator on X:

$$\Box_X=-\frac{1}{\cosh^2\alpha\sinh\alpha}\frac{\partial}{\partial\alpha}\cosh^2\alpha\sinh\alpha\frac{\partial}{\partial\alpha}+\frac{1}{\cosh^2\alpha}\Box_{S^2}-\frac{1}{\sinh^2\alpha}\Box_{S^1}. \tag{12}$$

If $y\in\Sigma$, then $\Box_{R^5}[rx,y]^{\mathrm{i}\mu-3/2}=0$. So

$$\Box_X[x,y]^{\mathrm{i}\mu-3/2}=(\mu^2+\tfrac{9}{4})[x,y]^{\mathrm{i}\mu-3/2}. \tag{13}$$

The spherical waves are given explicitly as

$$|k,l,m,\mu\rangle=\int_{S^2\times S^1}[x,y]^{\mathrm{i}\mu-3/2}Y_{lm}(\theta',\phi')\exp(\mathrm{i}k\Omega')\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi'\,\mathrm{d}\Omega' \tag{14}$$

where y,θ',ϕ',Ω' are as in (3). Set

$$|k,l,m,\mu\rangle=\Phi^{\mu}_{k,l}(\alpha)Y_{lm}(\theta,\phi)\exp(\mathrm{i}k\Omega) \tag{15}$$

where $\alpha,\theta,\phi,\Omega$ are as in (8). Then

$$\begin{aligned} &\Box_X|k,l,m,\mu\rangle=(\mu^2+\tfrac{9}{4})|k,l,m,\mu\rangle \\ &\Box_{S^2}|k,l,m,\mu\rangle=-l(l+1)|k,l,m,\mu\rangle \\ &\Box_{S^1}|k,l,m,\mu\rangle=-k^2|k,l,m,\mu\rangle \qquad \text{or} \qquad -m^2|k,l,m,\mu\rangle. \end{aligned} \tag{16}$$

Let

$$\Phi^{\mu}_{k,l}(\alpha)=\frac{1}{\cosh\alpha}W^{\mu}_{k,l}(\cosh\alpha) \tag{17}$$

then $W^{\mu}_{k,l}(z)$ satisfies the following equation:

$$(1-z^2)\frac{\mathrm{d}^2W}{\mathrm{d}z^2}-2z\frac{\mathrm{d}W}{\mathrm{d}z}-\left(\mu^2+\tfrac{1}{4}+\frac{k^2}{1-z^2}+\frac{l(l+1)}{z^2}\right)W=0. \tag{18}$$

Since the differential operator is self-adjoint

$$\langle k', l', m', \mu' | k, l, m, \mu\rangle$$

$$= \delta_{kk'}\delta_{ll'}\delta_{mm'} \int W_{kl}^{\mu'}(\cosh\alpha)\, W_{kl}^{\mu *}(\cosh\alpha)\sinh\alpha\, \mathrm{d}\alpha$$

$$= \delta_{kk'}\delta_{ll'}\delta_{mm'}\delta(\mu-\mu'). \qquad (19)$$

One recognises that $W_{k,0}^{\mu}(z)$ are the well known conical functions $\mathscr{P}^{k}_{i\mu-1/2}(z)$ discussed in the standard literature (Magnus *et al* 1966). The solution in general deserves further study in view of the importance of the group SO(3, 2).

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